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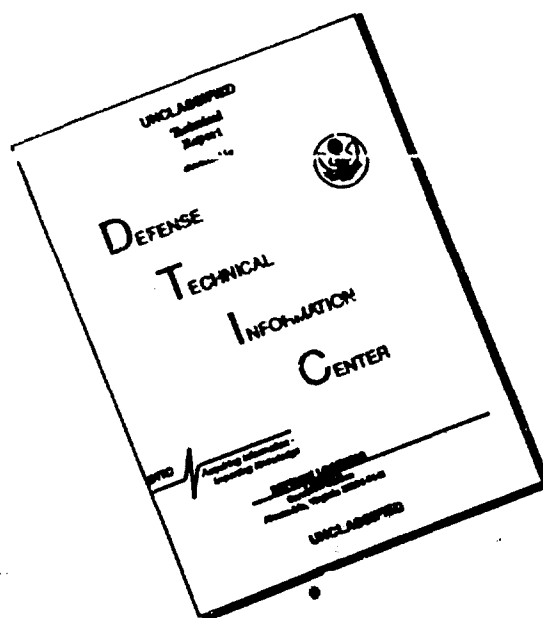
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Turbulent Diffusion in the Ground
Layer of the Atmosphere

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Turbulent Diffusion in the Ground Layer of the Atmosphere

A. S. Monin

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Solutions of the hyperbolic equation of turbulent flow in the near ground layer of air with indifferent stratification are investigated. As an example the diffusion of smoke at ambient temperature from a chimney is examined. The generalised expression for diffusion is indicated for cases with arbitrary temperature stratification of the air. ( )

Since the diffusion of a substance in a turbulent medium comes about because of turbulent fluctuations of wind velocity, the size of which is limited (and under the actual conditions in the lowest layers of the atmosphere is rather small) the propagation of the substance in the atmosphere takes place with a limited velocity and in a number of cases (e.g., after the release of an instantaneous point source) the propagation front of the substance can be observed. For describing such cases the common expression for diffusion having a parabolic character is shown to be inappropriate. In our earlier paper [1] a system is investigated in which allowance is clearly made for the final size of the turbulent fluctuations, and to describe turbulent diffusion we derive a system of equations of the hyperbolic type. The simplest version of such equations for the case of diffusion in the vertical direction z only, is of the form

$$\frac{\partial q_1}{\partial t} = a(q_2 - q_1) - \frac{\partial}{\partial z}uq_1, \quad \frac{\partial q_2}{\partial t} = a(q_1 - q_2) + \frac{\partial}{\partial z}uq_2, \dots (1)$$

where q_1 and q_2 are the probability densities for the co-ordinate z of the diffusing particles moving respectively upwards and downwards; t is time; u is the size of the turbulent fluctuations; z the velocity components; a is the frequency of change of direction of movement by the diffusing particles (frequency of "dispersion"). An analagous equation was proposed earlier in the work of V. A. Fok B. I. Davydov. and E. S. Lyapin quoted in reference [1] and also in the article by S. Goldstein [2] published in 1951, and is extremely close in content to Fok's work in 1926. Equations (1) correspond to the description of turbulent movements in the direction z as the aggregate of "streams" having flow rates u and -u, the diffusing particle transferring from streams of one type to streams of another type on an average with frequency a.

Examining only stationary turbulence we may consider the values u and a independent of time t. Introducing the density of the particles $q = q_1 + q_2$ and the density of the turbulent particle stream $Q = (q_1 - q_2)$ we reduce equations (1) to the form

$$\frac{\partial q}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + 2aQ = -u\frac{\partial}{\partial z}uq. \qquad \dots (2)$$

According to the first of these equations it is possible to put $q = \frac{\partial \Psi}{\partial z}$, $Q = -\frac{\partial \Psi}{\partial t}$ where $\Psi(z,t)$ will have the meaning of an integral function of probability distribution for the co-ordinate z of the diffusing particle. The second equation (2) takes the form of a telegraph equation (3):-

$$u\frac{\partial}{\partial z}u\frac{\partial \Psi}{\partial z} - \frac{\partial^2 \Psi}{\partial t^2} - 2a\frac{\partial \Psi}{\partial t} = 0. \qquad \dots (3)$$

The problem consists in the determination of the co-efficients of this equation $u(z)$ and $a(z)$ for different conditions in the atmosphere and in solving equation (3) with given initial and boundary conditions.

Paper [1] examined the case of diffusion in the field of homogeneous turbulence (u and a constants) in the lowest layer of the atmosphere with indifferent stratification. In the present paper we shall examine diffusion in the ground layer of the atmosphere with indifferent stratification in more detail than was done in reference [1]; we shall define the determination of the Riemann function for equation (3) in this case and examine an example of practical importance, i.e. the diffusion of smoke at ambient temperature from a chimney. Furthermore, we shall show how to determine the coefficients u(z) and a(z) for the ground layer of the atmosphere with arbitrary temperature stratification and analyse equation (3) for these conditions

1.   Diffusion in a Thermally Homogeneous Ground Layer of the Atmosphere

The stationary turbulent process in the ground layer of the atmosphere with indifferent stratification is characterised by a unique dimensional parameter - the friction velocity v^*, so that every parameter of turbulence having the dimensions of velocity must be proportional to v^*, the characteristic dimension of length must be proportional to height z, and the characteristic dimension of time must be proportional to $\frac{z}{v^*}$. Consequently in this case

$$u = \lambda v^*, \quad a = A\frac{v^*}{z}, \qquad \dots (4$$

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where λ and A are numerical constants. The constant λ should be determined from experiments as a ratio of the mean square value of the vertical fluctuations of velocity to $v*$; the experimental data in our arrangement shew that the value of λ is near to unity. The constant A may be determined by taking advantage of the steady-state solution of equations (2), which has the form

$$Q = \text{const}, \quad q(z_2) - q(z_1) = \frac{2AQ}{\lambda^2 v*} \log \frac{z_1}{z_2}.$$

We note that the coefficient $\frac{2AQ}{\lambda^2 v*}$ in the logarithmic formula for $q(z)$ must, as we know, equal $\frac{Q}{\varkappa v*}$ where $\varkappa$ is the Karman constant. Hence $A = \frac{\lambda^2}{2\varkappa}$. Introducing into equation (3) the dimensionless variables

$$\zeta = \frac{z}{h}, \quad \tau = \frac{\lambda v* t}{h}, \qquad \qquad \dots (5)$$

where h is some characteristic height, we may re-write this equation in the form

$$\frac{\partial^2 \Psi}{\partial \zeta^2} - \frac{\partial^2 \Psi}{\partial \tau^2} - \frac{2\varepsilon}{|\zeta|} \frac{\partial \Psi}{\partial \tau} = 0, \qquad \qquad \dots (6)$$

where $\varepsilon = \frac{\lambda}{2\varkappa}$ is a numerical constant. We are interested in the solution of this equation in the range $\zeta > 0$, satisfying the determined boundary condition at the earth's surface $\zeta = 0$. However, it is more convenient to consider not half the range $\zeta > 0$ but the whole range $-\infty < \zeta < \infty$, extending for this purpose the field of turbulence into the region $\zeta < 0$ with symmetry maintained with respect to the point $\zeta = 0$. If we assume that $a = \frac{\lambda^2 v*}{2\varkappa|z|}$, the corresponding coefficient in equation (6) is of the form $-\frac{2\varepsilon}{|\zeta|}$.

Let us restrict ourselves to seeking a solution of equation (6), corresponding to an instantaneous point source of diffusing substance at time $t = 0$ at the point $z = h$. The initial conditions for the original equations (1) will

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then take the form

$$q_1(z, 0) = \varepsilon_1 \delta(z - h), \quad q_2(z, 0) = \varepsilon_2 \delta(z - h),$$

where ε_1, ε_2 are the probability of positive and negative directions of initial movement by the diffusing particle. For example, particles of smoke being discharged from a chimney move upwards initially so that in this case $\varepsilon_1 = 1$, $\varepsilon_2 = 0$. With the explosion of a shell in the air, on the other hand, the particles as a rule have no preferred direction of initial movement; in this case we should assume that $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$. The initial conditions formulated for the function Ψ will thus be

$$\varphi(\zeta) = \Psi(\zeta, 0) = E(\zeta - 1), \quad \varphi(\zeta) = \left(\frac{\partial \Psi}{\partial \tau}\right)_{\tau=0} = (\varepsilon_2 - \varepsilon_1)(\zeta - 1), \quad \ldots.(7)$$

where $E(\zeta)$ is an improper distribution of probabilities, i.e., the function is equal to zero when $\zeta < 0$ and equal to unity when $\zeta \geq 0$.

Let us first examine the simplest case of a ground level source $(h = 0)$. In this case the function $\Psi(\zeta, \tau)$ can depend only on the ratio of its variables $\frac{\zeta}{\tau} = \xi = \frac{z}{\lambda v^* \tau}$, and equation (6) takes the form

$$(\xi^2 - 1)\frac{d^2 \Psi}{d\xi^2} + 2(\xi \mp \varepsilon)\frac{d\Psi}{d\xi} = 0$$

(the upper sign applies in the range $\zeta > 0$, the lower when $\zeta < 0$). Since the substance diffuses with a final velocity $u = \lambda v^*$, at the moment t it will be distributed according to the limits $-\lambda v^* t \leq z \leq \lambda v^* t$ so that $\Psi(\xi)$ can be distinguished from a constant only when $|\xi| < |$, and it must be that $\Psi(-1 - 0) = 0$, $\Psi(+ 1 + 0) = 1$. The first integral of the equation for $\Psi(\xi)$ takes the form

$$\frac{d\Psi}{d\xi} = \begin{cases} \dfrac{c_1}{1 - \xi^2}\left(\dfrac{1 - \xi}{1 + \xi}\right)^{\varepsilon} & \text{when } \zeta > 0, \\[3mm] \dfrac{c_2}{1 - \xi^2}\left(\dfrac{1 - \xi}{1 + \xi}\right)^{-\varepsilon} & \text{when } \zeta < 0, \end{cases}$$

where c_1, c_2 are constants of integration which must not be negative since the density of the diffusing substance $q(z,t) = \dfrac{d\Psi}{dz} = \dfrac{\Psi'(\zeta)}{\lambda v \bar{\mp} \Psi}$ is not negative. The expression for $\Psi(\zeta)$ which satisfies the condition. when $\zeta = \pm (1 + 0)$ has the form

$$\Psi(\zeta) = \begin{cases} 1 - \dfrac{c_1}{2\varepsilon}\left(\dfrac{1}{1} \bar{\mp} \left|\dfrac{\zeta}{\zeta}\right|\right)^\varepsilon & \text{when } 0 < \zeta \leq 1, \\[2ex] \dfrac{c_2}{2\varepsilon}\left(\dfrac{1}{1} \bar{\mp} \left|\dfrac{\zeta}{\zeta}\right|\right)^\varepsilon & \text{when } -1 \leq \zeta < 0. \end{cases}$$

At the point $\zeta = 0$ the function $\Psi(\zeta)$ has the discontinuity $\Psi(+0) - \Psi(-0) = 1 - \dfrac{c_1 + c_2}{2\varepsilon}$ the value of which has the meaning of that portion of the diffusing substance existing at the moment t at the earth's surface. This value does not vary with time and, naturally, may be equal either to unity or to zero. In the first case we get $c_1 = c_2 = 0$, i.e. $\Psi(\zeta) = E(\zeta)$ which denotes the absence of diffusion. Rejecting this uninteresting case we will assume $c_1 + c_2 = 2\varepsilon$. Then in the semi-distribution $\zeta < 0$ the portion of the diffusing substance $\dfrac{c_2}{2\varepsilon}$ exists the whole time not varying with time and consequently equal to its initial value ε_2. We thus get $c_2 = 2\varepsilon\varepsilon_2$, $c_1 = 2\varepsilon\varepsilon_1$. Finally we have

$$\Psi(\zeta) = \begin{cases} 1 - \varepsilon_1\left(\dfrac{1}{1} \bar{\mp} \left|\dfrac{\zeta}{\zeta}\right|\right)^\varepsilon & \text{when } 0 \leq \zeta \leq 1, \\[2ex] \varepsilon_2\left(\dfrac{1}{1} \bar{\mp} \left|\dfrac{\zeta}{\zeta}\right|\right)^\varepsilon & \text{when } -1 \leq \zeta \leq 0. \end{cases} \quad \ldots\ldots(8)$$

Here $\Psi(\zeta)$ is continuous at the point $\zeta = 0$ (since $\varepsilon_1 + \varepsilon_2 = 1$) but the density of the diffusing substance $q(z,t)$ undergoes a perturbation:

$$q(-0, t) = \dfrac{2\varepsilon\varepsilon_2}{\lambda v*t}, \quad q(+0, t) = \dfrac{2\varepsilon\varepsilon_1}{\lambda v \bar{\mp} t},$$

The existence of this discontinuity is reasonable since diffusion of the substance at the point $\zeta = 0$, at which the frequency of dispersion $a = \frac{\lambda^2 v^*}{2\varkappa |z|}$ is infinitely great, is extremely difficult. By analogy the absence of discontinuities in the function $\Psi(\zeta,\tau)$ at the "fronts" $|\zeta| = \tau$ is explained, i.e. the equality to zero of the probabilities of finding the diffusing particles at points $z = \pm \lambda v^* t$. Indeed each particle discharged from a source $z = 0$ almost certainly undergoes dispersion while still near the source and consequently does not reach the "front".

When $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ all the diffusing substance at any moment in time is in the half range $z > 0$ so that this case corresponds to the condition of "reflection" of the substance from the earth's surface. Then

$$q(z, t) = \frac{1}{\varkappa v^* t} \frac{(1 - \frac{z}{\zeta})^{\varepsilon-1}}{(1 + \frac{z}{\zeta})^{\varepsilon+1}}, \quad 0 \leq z \leq \lambda v^* t. \qquad \dots (9)$$

The character of distribution of the substance in height depends here essentially on the number $\varepsilon = \frac{\lambda}{2\varkappa}$, i.e. on the ratio λ, the size of the vertical fluctuations of wind speed to the friction velocity. Theoretically there are two conceivable forms of "front" for the diffusion of the substance. If $\lambda > 2\varkappa \approx 0.8$, i.e., if $\varepsilon > 1$, then when approaching the "front" the concentration of the substance tends towards zero (since the frequency of dispersion $a = \frac{\lambda^2}{2\varkappa} \frac{v^*}{z}$ is then comparatively small), so that the "front" does not appear as a surface of sharp discontinuity. But if $\lambda < 2\varkappa$, i.e. $\varepsilon < 1$, then when approaching the "front" the concentration increases (since the frequency of dispersion under these conditions is comparatively small), so that the "front" appears as a surface of sharp discontinuity. Under actual conditions in the ground layer of the atmosphere it appears that the first of these cases is realised, i.e. the value $\varepsilon > 1$ applies.

In the presence of a continuous line source of diffusing substance perpendicular to the wind direction, and neglecting horizontal dispersion in the wind direction, the formula for the concentration is found by replacing time t in (9) by $\frac{x}{U}$, where x is the co-ordinate along the wind direction and U is the wind speed, considered constant. We thus get

$$q(x, z) = \frac{c}{x v^* x} \frac{\left(1 - \frac{Uz}{\lambda v^* x}\right)^{\varepsilon - 1}}{\left(1 + \frac{Uz}{\lambda v^* x}\right)^{\varepsilon + 1}}, \quad 0 \leq z \leq \frac{\lambda v^* x}{U}.$$

The vertical profiles of concentrations at various distances x from the source are seen to resemble one another and their form is characterised by the function $f(\zeta) = (1 - \zeta)^{\varepsilon - 1}(1 + \zeta)^{-\varepsilon - 1}$ the graph of which with values $\varepsilon = \frac{3}{4}, 1, \frac{5}{4}$ is shown in Figure 1.

2. Diffusion from an Elevated Source

The solution of equation (6) under arbitrary initial conditions $\Psi(\zeta, 0) = \varphi(\zeta), \left(\frac{\partial \Psi}{\partial \tau}\right)_{\tau = 0} = \phi(\zeta)$ is supplied by the Riemann formula, which for the given equation has the form

$$2\Psi(\zeta, \tau) = \varphi(\zeta - \tau)v(\zeta, \tau; \zeta - \tau, 0) + \varphi(\zeta + \tau)v(\zeta, \tau; \zeta + \tau, 0)$$
$$+ \int_{\zeta - \tau}^{\zeta + \tau} \left\{ \phi(\zeta')v(\zeta, \tau; \zeta', 0) + \varphi(\zeta')\left[\frac{2\varepsilon}{|\zeta'|}v(\zeta, \tau; \zeta', 0) \right.\right.$$
$$\left.\left. - \frac{\partial v(\zeta, \tau; \zeta', 0)}{\partial \tau} \right] \right\} d\zeta'. \qquad \qquad \dots (10)$$

Here $v(\zeta, \tau; \zeta', \tau')$ is the Riemann function, characterising the influence of a single "impulse" applied at the point ζ' at moment τ' which develops at the point ζ at moment $\tau > \tau'$. In accordance with equation (6) the disturbances propagate with unit (dimensionless) speed, so that the Riemann function differs from zero only when $\tau - \tau' \geq |\zeta - \zeta'|$. As a consequence of the independence of the coefficients of the equation and time, it (the Riemann function) depends only on the difference of the variables τ and τ'. Moreover, according to its meaning it cannot depend on the value of h, which enters into the scales of measurement of length and time, i.e., it can only depend on the ratios of its variables. Consequently, the Riemann function in our case must take the form

$$v(\zeta, \tau; \zeta', \tau') = V\left(\frac{\zeta}{\tau - \tau'}, \frac{\zeta'}{\tau - \tau'} \right);$$

where $V(\zeta, \zeta')$ differs from zero only when $|\zeta - \zeta'| \leq 1$.

Moreover, from considerations of symmetry it must be that $V(-\zeta, \zeta') = V(\zeta, -\zeta')$.

For ζ' and τ' the Riemann function satisfies an equation linked to (6) i.e. obtainable from (6) by substituting ζ', τ' for ζ, τ and changing the sign before the last term. For $V(\zeta, \zeta')$ this expression takes the form

$$\frac{\partial^2 V}{\partial \zeta'^2} - \left(\zeta \frac{\partial}{\partial \zeta} + \zeta' \frac{\partial}{\partial \zeta'} + 1 - \frac{2\varepsilon}{|\zeta'|} \right) \left(\zeta \frac{\partial V}{\partial \zeta} + \zeta' \frac{\partial V}{\partial \zeta'} \right) = 0.$$

For $\zeta \neq \zeta'$ and $\tau \neq \tau'$ when ζ and τ the Riemann function satisfies equation (6) giving for $V(\zeta, \zeta')$ an equation differing from that obtained above by alteration of the positions of ζ and ζ'. The difference of these two equations has the form

$$\frac{\partial^2 V}{\partial \zeta^2} - \frac{\partial^2 V}{\partial \zeta'^2} + 2\varepsilon \left(\frac{1}{|\zeta|} - \frac{1}{|\zeta'|} \right) \left(\zeta \frac{\partial V}{\partial \zeta} + \zeta' \frac{\partial V}{\partial \zeta'} \right) = 0. \quad \ldots (11)$$

The required solution of this equation is clearly determined by the known conditions in the characteristics of equation (6) i.e. in straight lines $\tau - \tau' = \pm (\zeta - \zeta')$ or $\zeta - \zeta' = \pm 1$.

The indicated conditions for the function $V(\zeta, \zeta')$ take the form

$$\frac{d \log V(\zeta, \zeta + 1)}{d\zeta} = \frac{\varepsilon}{\zeta |\zeta + 1|}, \quad \frac{d \log V(\zeta + 1, \zeta)}{d\zeta} = \frac{\varepsilon}{|\zeta|(\zeta + 1)},$$

whence are obtained the following values of the Riemann function for the characteristics

$$V(\zeta + 1, \zeta) = V(\zeta, \zeta + 1) = \begin{cases} \left(\dfrac{\zeta}{\zeta + 1} \right)^{\varepsilon} & \text{when } \zeta > 0, \\ \left(\dfrac{\zeta + 1}{\zeta} \right)^{\varepsilon} & \text{when } \zeta < -1, \\ 0 & \text{when } -1 \leq \zeta \leq 0. \end{cases}$$

$$\ldots (12)$$

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Hence, in particular, it follows that the function $V(\xi, \xi')$ is symmetrical with respect to its variables.

The particular solution (8) found above of equation (6) immediately permits determination of the values of the Riemann function $V(\xi, \xi')$ when $\xi' = 0$. Actually, substituting in the Riemann formula (10) the initial conditions (7) we get after simple re-arrangement with $\zeta < 1 + \tau$:

$$V(\zeta, \tau) = \tfrac{1}{2}E(\zeta - \tau - 1)V\left(\tfrac{\zeta}{\tau}, \tfrac{\zeta}{\tau} - 1\right) + \tfrac{1}{2}E(\zeta + \tau - 1)V\left(\tfrac{\zeta}{\tau}, \tfrac{\zeta}{\tau} + 1\right)$$

$$+ \tfrac{\varepsilon_2 - \varepsilon_1}{2}V\left(\tfrac{\zeta}{\tau}, \tfrac{1}{\tau}\right) + \tfrac{1}{2}\int_{\frac{\zeta}{\tau} - 1}^{\frac{\zeta}{\tau} + 1} E\left(\zeta' - \tfrac{1}{\tau}\right)\left[\tfrac{2\varepsilon}{1 \xi'}V\left(\tfrac{\zeta}{\tau}, \xi'\right)\right.$$

$$\left. - \tfrac{\zeta}{\tau}\frac{\partial V\left(\tfrac{\zeta}{\tau}, \xi'\right)}{\partial \xi} - \xi'\frac{\partial V\left(\tfrac{\zeta}{\tau}, \xi'\right)}{\partial \xi'}\right]d\zeta'. \qquad \dots(13)$$

Proceeding in this formula to the limit where $h \to 0$ we are satisfied that $V(\xi, 0)$ is the item in solution (8) corresponding to a ground instantaneous point source of the admixture, which has the coefficient $\tfrac{\varepsilon_2 - \varepsilon_1}{2}$. Eliminating this item in formula (8) we obtain

$$V(\xi, 0) = V(0, \xi) = \left(\frac{1 - \left|\frac{\xi}{\xi}\right|}{1 + \left|\frac{\xi}{\xi}\right|}\right)^{\varepsilon}. \qquad \dots(14)$$

The complete expression for the Riemann function will be sought in the form

$$V(\xi, \xi') = \left|\frac{|\xi| + |\xi'| - 1}{|\xi| + |\xi'| + 1}\right|^{\varepsilon} W(Z), \quad Z = \frac{1 - \frac{(\xi - \xi')^2}{4\xi\xi'}}{} \qquad \dots(15)$$

This expression possesses the requisite properties of symmetry $V(\xi, \xi') = V(\xi', \xi) = V(-\xi, -\xi')$. By the characteristics $\xi - \xi' = \pm 1$ the first factor is transformed into expression (12), while the variable Z of the function W is zero, so that $W(0) = 1$. The area of determination

$|\xi, \xi'| \le 1$ of the function $V(\xi, \xi')$ naturally breaks into three parts in which the function $W(Z)$ can be determined by difference (Figure 2.). In the first place it allows us to distinguish the areas in which ξ and ξ' have the same signs, from areas in which the signs of ξ and ξ' are different. In the second place, for the characteristic $\xi' - \xi = 1$ at points $\xi = 0$ and $\xi = -1$ the Riemann function is not analytical and since the properties are distributed along the characteristics, it is possible to anticipate breaking of the analytical ability of the Riemann function at the characteristics $\xi' + \xi = \pm 1$ which separate at the part of the area where the signs of ξ and ξ' are alike. Therefore we consider three zones

$$\text{I} \quad |\xi - \xi'| \le 1 \le |\xi + \xi'|,$$
$$\text{II} \quad |\xi + \xi'| \le 1, \ \xi\xi' \ge 0,$$
$$\text{III} \quad |\xi - \xi'| \le 1, \ |\xi + \xi'| \le 1, \ \xi\xi' \le 0.$$

Substituting (15) in (11), in zone I we obtain for $W(Z)$ the equation

$$Z(Z - 1)^2 W'' + (Z - 1)(2Z - 1)W' - \varepsilon^2 W = 0.$$

In addition it is necessary to choose a solution of the equation, satisfying the condition $W(0) = 1$. In zone II for $W(z)$ such an equation is obtained and it is necessary to choose a solution bordering on the first zone $(Z = 1)$ coincident with the solution for the first zone, and moreover satisfying the condition $W(\infty = 1)$ emerging from comparison of formulae (14) and (15). In zone III for $W(Z)$ the equation obtained is·

$$Z^2(Z - 1)W'' + Z(2Z - 1)W' + \varepsilon^2 W = 0,$$

the solution of which must satisfy the conditions $W(-\infty) = 1$ and $W(Z) = 0(Z^{-\varepsilon})$ with $Z \to -0$. Using the formulae for the analytical continuation of the hypergeometric functions we obtain

$$(1 - Z)_2^\varepsilon F_1(\varepsilon, 1 + \varepsilon; 1; Z) \text{ in zone I,}$$

$$W(Z) = \begin{cases} \left[\frac{\Gamma^2(1 + \varepsilon)}{\Gamma(1 + 2\varepsilon)} - \frac{1}{2}\frac{\Gamma(1 - 2\varepsilon)}{\Gamma^2(1 - \varepsilon)}\right]\left(\frac{Z - 1}{Z}\right)^\varepsilon {}_2F_1\left(\varepsilon, \varepsilon; 1 + 2\varepsilon; \frac{Z - 1}{Z}\right) \\ \quad + \frac{\Gamma(2\varepsilon)}{\Gamma(\varepsilon)\Gamma(1 + \varepsilon)}\left(\frac{Z - 1}{Z}\right)^{-\varepsilon} {}_2F_1\left(-\varepsilon, -\varepsilon; 1 - 2\varepsilon; \frac{Z - 1}{Z}\right) \text{ in zone II,} \\ \frac{\Gamma^2(1 + \varepsilon)}{\Gamma(1 + 2\varepsilon)}(-Z)^\varepsilon {}_2F_1\left(-\varepsilon, -\varepsilon; 1 - 2\varepsilon; \frac{Z - 1}{Z}\right) \text{ in zone III.} \end{cases}$$

$$....(16)$$

Here $_2F_1(\alpha, \beta, \gamma; Z)$ is the symbol of the hypergeo-metric function. Formulae (15) - (16) specify the deter-mination of the Riemann function quoted in reference [1]. Together with formula (13) they give a solution of equation (6) for the initial conditions (7). According to (13) this solution with $\zeta < 1 - \tau$ is equal to zero; at the point $\zeta = 1 - \tau$ (the lower diffusion "front" of the substance) it has a discontinuity of value $\varepsilon_2 V\left(\frac{1}{\tau} - 1, \frac{1}{\tau}\right)$; in the interval $1 - \tau \leq \zeta < 1 + \tau$ it is determined by the formula

$$\Psi(\zeta, \tau) = \frac{1}{2}V\left(\frac{\zeta}{\tau}, \frac{\zeta}{\tau} + 1\right) + \frac{\varepsilon_2 - \varepsilon_1}{2}V\left(\frac{\zeta}{\tau}, \frac{1}{\tau}\right)$$

$$+ \frac{1}{2}\int_{1/\tau}^{\frac{\zeta}{\tau}+1}\left[\frac{2\varepsilon}{\xi'\tau}V\left(\frac{\zeta}{\tau}, \xi'\right) - \frac{\zeta}{\tau}\frac{\partial V\left(\frac{\zeta}{\tau}, \xi'\right)}{\partial\xi} - \xi'\frac{\partial V\left(\frac{\zeta}{\tau}, \xi'\right)}{\partial\xi}\right]d\xi',$$

while at the point $\zeta = 1 + \tau$ (the upper diffusion "front" of the substance) it has a discontinuity of value $\varepsilon_1 V\left(\frac{1}{\tau} + 1, \frac{1}{\tau}\right)$ and with $\zeta \geq 1 + \tau$ is equal to unity.

The latter is demonstrated in the following way. When $\zeta \geq 1 + \tau$ formula (13) is superseded by the formula

$$\Psi(\zeta, \tau) = \frac{1}{2}V\left(\frac{\zeta}{\tau}, \frac{\zeta}{\tau} - 1\right) + \frac{1}{2}V\left(\frac{\zeta}{\tau}, \frac{\zeta}{\tau} + 1\right)$$

$$+ \frac{1}{2}\int_{\zeta/\tau-1}^{\zeta/\tau+1}\left[\frac{2\varepsilon}{\tau\xi'}V\left(\frac{\zeta}{\tau}, \xi'\right) - \frac{\zeta}{\tau}\frac{\partial V\left(\frac{\zeta}{\tau}, \xi'\right)}{\partial\xi} - \xi'\frac{\partial V\left(\frac{\zeta}{\tau}, \xi'\right)}{\partial\xi'}\right]d\xi'.$$

Differentiating this equation for τ and using conditions (12) and equation (11), we are convinced that $\frac{\partial \Psi(\zeta, \tau)}{\partial \tau} \equiv 0$. At the same time $\Psi(\zeta, 0) \equiv 1$ when $\zeta \geq 1$. Consequently $\Psi(\zeta, \tau)$ when $\zeta \geq 1 + \tau$ similarly equals unity.

So far we have not touched upon the question of extreme conditions for concentration of the diffusing substance at the earth's surface. When $\tau < 1$ this question does not arise, since the substance issuing from the source has not yet reached the earth's surface. The solution of (13) is here the probability distribution. The presence of rapid changes in the function $\Psi(\zeta, \tau)$ at the points $\zeta = 1 + \tau$ implies that at the "fronts", which are being propagated from the source upwards and downwards, there exists a finite quantity of substance: at the "front" which is moving upwards there is a proportion of the substance $\varepsilon_2(1 - \tau)^{-\varepsilon}$, diminishing with time according to a power law; at the "front" which is moving downwards, there is a proportion of the substance $\varepsilon_2(1 - \tau)^{\varepsilon}$, diminishing to zero with the approach of the "front" towards the earth's surface (progress of the particles towards the earth's surface is difficult since the frequency of distribution is increasing here without restriction).

When $\tau > 1$ the solution obtained does not show the probability of distribution and consequently does not describe in any way the distribution of the substance. For proof of this it is sufficient to determine the value $\Psi(0, \tau)$ when $\tau > 1$ which is completely specified by the function $V(0, \zeta') = V(\zeta', 0)$. Applying (13) and (14) convinces us that $\Psi(0, \tau) \to \infty$ with $\tau \to \infty$, which proves the statement put forward. It is possible to shew that equation (6) has no solution describing the distribution of the substance for an instantaneous point source, for which the diffusion of the substance would be through the particular point $\zeta = 0$ or for material accumulating at that point. This situation is completely analagous to the position for a parabolic equation of diffusion with diffusion coefficient $k(z) = \kappa v^* z$ which also has no solutions describing the distribution of the substance under the conditions of complete or partial absorption of substance at the point $z = 0$. At the same time we can obtain a physically understandable solution of equation (6) when $\tau > 1$ by introducing the condition of reflection of the diffusing particles from the earth's surface. For that it is necessary to add to the solution of (13) a solution corresponding to the virtual source of diffusing substance at the point $z = -h$, i.e. satisfying the initial conditions

$$q_1(z, 0) = \varepsilon_2\delta(z + h), \quad q_2(z, 0) = \varepsilon_1\delta(z + h),$$

which can be re-written in the form

$$\varphi(\zeta) = E(\zeta + 1), \quad \psi(\zeta) = (\varepsilon_1 - \varepsilon_2)\delta(\zeta + 1).$$

Applying considerations of symmetry, it is easy to see that under conditions of reflection the probability distribution differing from zero only when $\zeta > 0$, should be determined by the formula $\Psi(\zeta, \tau) - \Psi(-\zeta, \tau)$ where $\Psi(\zeta, \tau)$ is the function determined above. With $\tau \leq 1$ this distribution does not differ from $\Psi(\zeta, \tau)$ but with $\tau > 1$ a difference will only occur in the region $0 \leq \zeta \leq \tau - 1$ in which the influence of a "reflected wave" is felt. At the same time the portion of the diffusing substance at the "front of the reflected wave" equals zero.

The results obtained permit us to make calculations relating to the solution of concrete problems. As an example let us calculate the concentration of the substance at the earth's surface with an instantaneous point source at height h discharging all the substance upwards. Assuming that the substance is not absorbed by the earth's surface and applying the preceding reasoning, we must suppose (when $\tau \geq 1$) that:-

$$q(0, t) = \frac{1}{h}\left[\frac{\partial \Psi(+0, \tau)}{\partial \zeta} + \frac{\partial \Psi(-0, \tau)}{\partial \zeta}\right],$$

where $\Psi(\zeta, \tau)$ is determined by formula (13). The last expression can be put into the form

$$q(0, t) = \frac{1}{2h\tau}\left[2\varepsilon \int_{1/\tau}^{1} f(\xi')\frac{d\xi'}{\xi'} - \frac{\tau - 1}{\tau}f\left(\frac{1}{\tau}\right)\right],$$

where

$$f(\xi') = \frac{\partial V(+0, \xi')}{\partial \zeta} + \frac{\partial V(-0, \xi')}{\partial \zeta} = 4\varepsilon\xi'\frac{\left(1 - |\xi'|\right)^{\varepsilon - 1}}{\left(1 + |\xi'|\right)^{\varepsilon + 1}}.$$

After simple calculations we obtain

$$q(0, t) = \frac{2\varepsilon}{h}\frac{(\tau - 1)^{\varepsilon}}{(\tau + 1)^{\varepsilon + 1}}, \quad \tau \geq 1. \qquad \ldots(17)$$

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With $h \to 0$ the same result is obtained from this formula as by formula (9), as it should be.

Let us consider a chimney of height h uniformly discharging smoke at neutral temperature with velocity c g per unit time. Assuming the smoke is finely divided (so that its gravitational settling can be ignored), neglecting changes in wind speed U with height and turbulent mixing along the wind direction we may calculate the concentration according to the formula

$$q(x, y, z) = c \int_0^\infty \delta(x - Ut) \frac{e^{-\frac{y^2}{4kt}}}{2\sqrt{\pi kt}} \frac{1}{h} \left[\frac{\partial \Psi(\zeta, \tau)}{\partial \zeta} - \frac{\partial \Psi(-\zeta, \tau)}{\partial \zeta} \right] dt.$$

Here x, y are the horizontal co-ordinates (x is direction of wind), and k the coefficient of horizontal mixing in the direction y. In particular, applying (17), we obtain a formula for the concentration from the chimney at the earth's surface in the direction of the wind:

$$q(x, 0, 0) = \frac{c\varepsilon}{Uh} \sqrt{\frac{\lambda v^*}{\pi kh}} \frac{1}{\sqrt{\zeta}} \frac{(\zeta - 1)^\varepsilon}{(\zeta + 1)^{\varepsilon + 1}}, \quad \zeta = \frac{\lambda v^* x}{Uh} \geq 1. \quad \dots (18)$$

The abscissa of the maximum of this function is proportional to h, and the value of the maximum is inversely proportional to $h^{3/2}$. Figure 3 shows the graph of the universal function $\zeta^{-\frac{1}{2}}(\zeta - 1)^\varepsilon (\zeta + 1)^{-\varepsilon - 1}$ when $\varepsilon = \frac{3}{4}$, 1, $\frac{5}{4}$, characterising the form of the concentration distribution at the earth's surface in the wind direction.

Let us now briefly consider the case where the diffusing substance is absorbed by the earth's surface. In this case the extreme condition q = 0 should be formulated not with z → 0 but with z = z_0 (roughness length). Reckoning the height z from the plane z_0 we must suppose $a = \frac{\lambda^2}{2\varkappa} \frac{v^*}{z_0 + |z|}$, so that the function $\Psi(\zeta, \tau)$ must be determined from equation:

$$\frac{\partial^2 \Psi}{\partial \zeta^2} - \frac{\partial^2 \Psi}{\partial \tau^2} - \zeta_0 \frac{2\varepsilon}{\zeta_0 + |\zeta|} \frac{\partial \Psi}{\partial \tau} = 0, \quad \zeta_0 = \frac{z_0}{h} > 0. \quad \dots (19)$$

The Riemann function of this equation will take the form

$$v\left(\frac{\zeta}{\zeta_o}, \frac{\zeta'}{\zeta_o}, \tau - \frac{\tau'}{\zeta_o}\right).$$

The determination of this function and solution of equation (19) under the given initial and extreme conditions can be achieved in an analogous manner to that stated above.

3. Diffusion in a Thermally Heterogeneous Ground Layer of Air

A turbulent system in a thermally heterogeneous ground layer of air is characterised by three dimensional parameters: v^*, $\frac{q}{c_p\rho}$ and $\frac{g}{T_o}$, where q is the vertical turbulent stream of heat; c_p and ρ, the specific heat and density of air; g, the accelerating force of gravity; and T_o, the normal temperature of the ground layer of air (see for example [3]). From the indicated parameters it is possible to form a length scale

$$L = \frac{(v^*)^3}{\varkappa \frac{g}{T_o}\left(-\frac{q}{c_p\rho}\right)}$$

In accordance with considerations of the dimensional parameter of turbulence u and a, occurring in the equation of diffusion (3), it is possible to write them in the form

$$u = \lambda v^* \varphi_1\left(\frac{z}{L}\right), \quad a = \frac{\lambda^2}{2\varkappa}\frac{v^*}{z}\varphi_2\left(\frac{z}{L}\right), \quad \ldots (20)$$

where $\varphi_1(\zeta)$ and $\varphi_2(\zeta)$ are certain universal functions. In the particular case of indifferent stratification of the air scale L becomes infinitely large while the variable $\frac{z}{L}$ of the functions φ_1 and φ_2 becomes zero, so that for agreement with formula (4) (corrected for a system with indifferent stratification) we must assume $\varphi_1(0) = \varphi_2(0) = 1$.

It is possible to determine the function $\varphi_1(\zeta)$ by making use of the equilibrium equation of turbulent energy (see for example [4]) which for the coefficient of turbulence k leads to the expression

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$$k = v^*1\left(1 - \frac{Ri}{Ri_{xp}}\right)^{\frac{1}{4}} \qquad \dots (21)$$

where 1 is the scale of turbulence (mixing length in the Prandtl sense), Ri is the Richardson number and Ri_{cr} its critical value. The value $\frac{k}{1}$ is proportional to the mean square velocity of the turbulent fluctuations, i.e. $\frac{k}{1} \sim \mu$. Hence, applying the condition $\varphi_1(0) = 1$, we obtain

$$\varphi^1(\zeta) = \left[1 - \frac{Ri(\zeta)}{Ri_{xp}}\right]^{-\frac{1}{4}}. \qquad \dots (22)$$

In order to determine the function $\varphi_2(\zeta)$, we examine the steady-state solution of equations (2), which arises from the conditions

$$Q = const, \quad 2aQ = -u\frac{\partial}{\partial z}uq.$$

The right-hand side of the last equation may be re-written in the approximate form $-u^2\frac{\partial q}{\partial z}$, so that the function $u(z)$ changes significantly more slowly than $q(z)$. The idea of this simplification is to ignore the vertical stream of diffusing substance which occurs when the concentration gradient of the substance is zero and which is a consequence of a difference in the intensity of turbulent fluctuations of velocity in different layers of air (the presence of such a stream of substance is analogous to the effect of thermal diffusion). With the indicated simplification, by applying (20) we can write a steady-state solution of equations (2) in the form

$$\frac{\partial q}{\partial z} = -\frac{Q}{xv^*z} \frac{\varphi_2(z/L)}{\varphi_1^2(z/L)}.$$

According to the theory of similarity for a stationary turbulent system in a thermally heterogeneous ground layer of air (see for example reference [3]) it must be that

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$$q(z_2) - q(z_1) = -\frac{Q}{\varkappa v^*}\left[f\left(\frac{z_2}{L}\right) - f\left(\frac{z_1}{L}\right)\right],$$

where $f(\zeta)$ is a universal function which is linked with the Richardson number by the relation

$$\frac{Ri}{Ri_{\varkappa p}} = -\frac{1}{f'\left(\frac{z}{L}\right)}.$$

Consequently we obtain

$$\varphi_1(\zeta) = \left[1 - \frac{1}{f'(\zeta)}\right]^{\frac{1}{4}}, \quad \varphi_2(\zeta) = \zeta f'(\zeta)\left[1 - \frac{1}{f'(\zeta)}\right]^{\frac{1}{2}},$$

$$\cdots\cdots(23)$$

and equation (3), by calculations using equations (20) and (23), becomes

$$\left[1 - \frac{1}{f'(\zeta)}\right]^{\frac{1}{4}}\frac{\partial}{\partial\zeta}\left[1 - \frac{1}{f'(\zeta)}\right]^{-\frac{1}{4}}\frac{\partial\gamma}{\partial\zeta} - \frac{\partial^2\gamma}{\partial\zeta^2} - 2\zeta f'(\zeta)\left[1 - \frac{1}{f'(\zeta)}\right]^{\frac{1}{2}}\frac{\partial\gamma}{\partial\tau} = 0, \quad\cdots(24)$$

when the dimensionless variables $\zeta = \frac{z}{L}$, $\tau = \frac{\varkappa v^* t}{L}$ are introduced.

For the analysis of this equation it is essential to know the nature of the function $f'(\zeta)$. We use the equations of motion and flow of heat for a steady-state system in the ground layer of air, which have the form

$$k\frac{dU}{dz} = (v^*)^2, \quad \alpha k\frac{dT}{dz} = -\frac{q}{c_p\rho}.$$

Here U and T are the average velocity and temperature, and $\alpha = \frac{1}{Ri_{cr}}$. From these equations and determination of the Richardson number we obtain $k = \varkappa v^* L/f'(\zeta)$. Substituting this formula in (21) and determining l (in the case of stable stratification to which our present discussion is confined)

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we find from the formula

$$1 = \varkappa z \left(1 - \frac{Ri}{Ri_{\varkappa p}}\right)^{3 - \frac{1}{4}},$$

[as used in the paper [4] (in which for concurrence with experimental data we should put $\beta = \frac{1}{5}$)] for $f'(\zeta)$ the algebraic expression

$$\frac{1}{f'(\zeta)} = \zeta\left[1 - \frac{1}{f'(\zeta)}\right]^{3/5}.$$

From this equation it follows that with $\zeta \ll 1$ then $f'(\zeta) = \frac{1}{\zeta} + \frac{3}{5} + O(\zeta)$ and with $\zeta \to \infty$ asymtotically $f'(\zeta) \sim 1 + \zeta^{-5/3}$ to which the following limiting formulae correspond for the functions (23)

when $\zeta \ll 1$ $\varphi_1(\zeta) \approx 1 - \frac{1}{4}\zeta$, $\varphi_2(\zeta) \approx 1 + \frac{1}{10}\zeta$,

when $\zeta \to \infty$ $\varphi_1(\zeta) \sim \zeta^{5/12}$, $\varphi_2(\zeta) \sim \zeta^{5/6}$.

Thus for small values of ζ the functions φ_1 and φ_2 are extremely close to unity and the equation for Υ takes the same form as in the case of indifferent stratification. With $\zeta \to \infty$ the fluctuation velocity (and consequently the velocity of dispersion of the substance) decreases as $\zeta^{-5/12}$, while the frequency of dispersion decreases as $\zeta^{-\frac{5}{6}}$; the "coefficient of diffusion" $\frac{u^2}{2a}$ tends towards a constant value.

In order to establish the parameters of equation (24), we introduce instead of ζ a new independent variable

$$\eta = \int_0^\zeta \left[1 - \frac{1}{f'(\zeta)}\right]^{-\frac{1}{4}} d\zeta = \frac{8}{3}\left\{1 - \left[1 - \frac{1}{f'(\zeta)}\right]^{3/20}\right\} - \frac{10}{17}\left\{1 - \left[1 - \frac{1}{f'(\zeta)}\right]^{-17/20}\right\}, \quad ....(25)$$

which for small values of ζ has the form $\eta \sim \zeta$, and for large values of ζ grows as ζ^{20}. Then equation (24) will be written in the form

$$\frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \tau^2} - 2\varepsilon g(\eta)\frac{\partial \psi}{\partial \tau} = 0. \qquad \ldots (26)$$

Moreover, the function $g(\eta)$ is parametrically represented by the formulae

$$g(\eta) = \frac{\sqrt{x}}{1 - x}, \quad \eta = \frac{6}{5}(1 - x^{5/20}) - \frac{12}{17}(1 - x^{-17/20}), \quad 0 \le x \le 1.$$

For small values of η, $g(\eta) \approx \frac{1}{\eta}$, and for large values of η asymptotically $g(\eta) \sim \left(\frac{17}{12}\eta\right)^{-20/17}$. A graph of the function $g(\eta)$ is shown in Figure 4.

Integration of equation (26) is possible by the use of methods of approximation.

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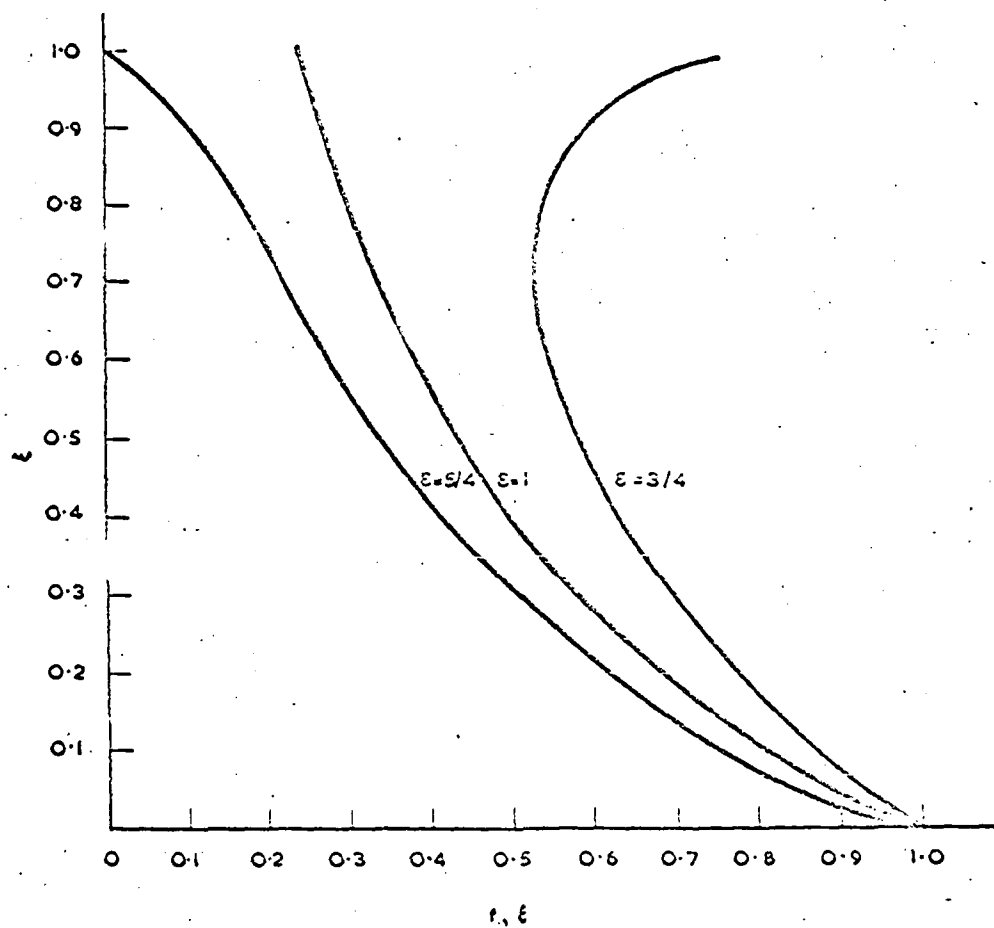
FIGURE 1.   THE SHAPE OF THE VERTICAL PROFILES OF CONCENTRATION FOR A
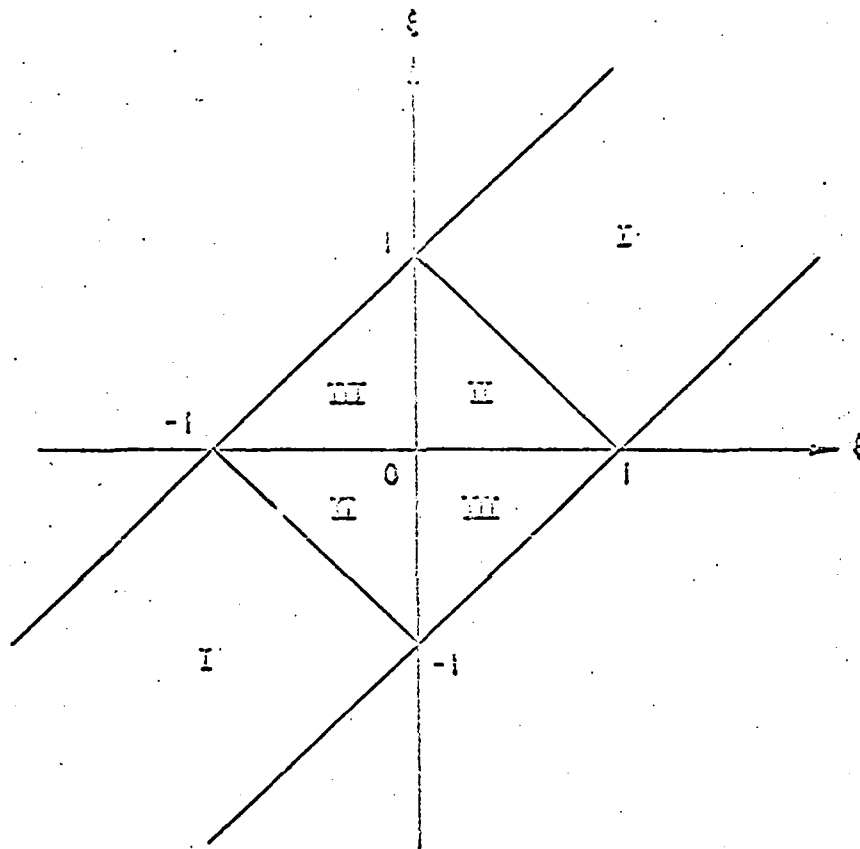CONTINUOUS LINE SOURCE AT GROUND LEVEL
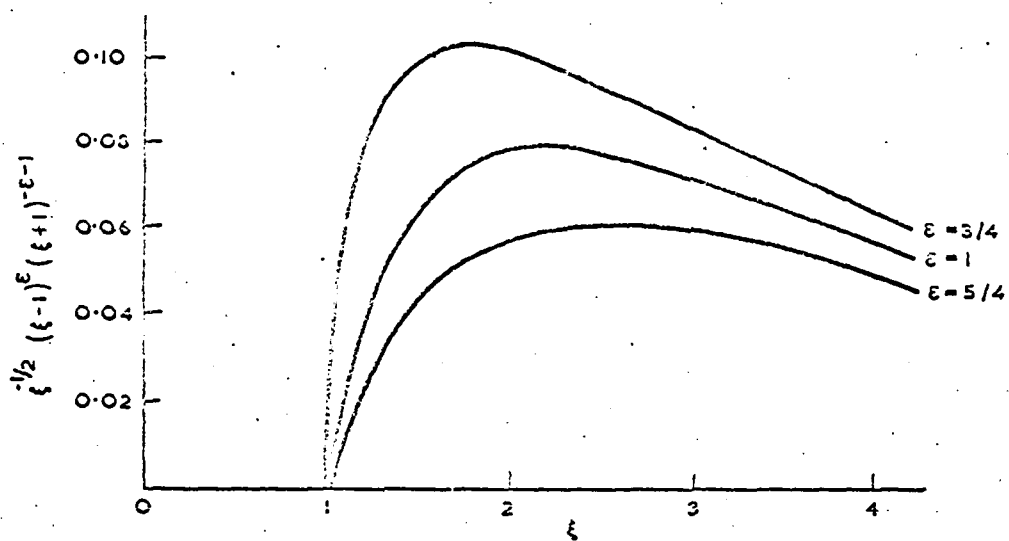
FIGURE 2. AREA OF DETERMINATION OF THE RIEMANN FUNCTION

FIGURE 3.   THE CONCENTRATION OF SMOKE AT THE EARTH'S SURFACE IN THE WIND DIRECTION, FROM A CHIMNEY
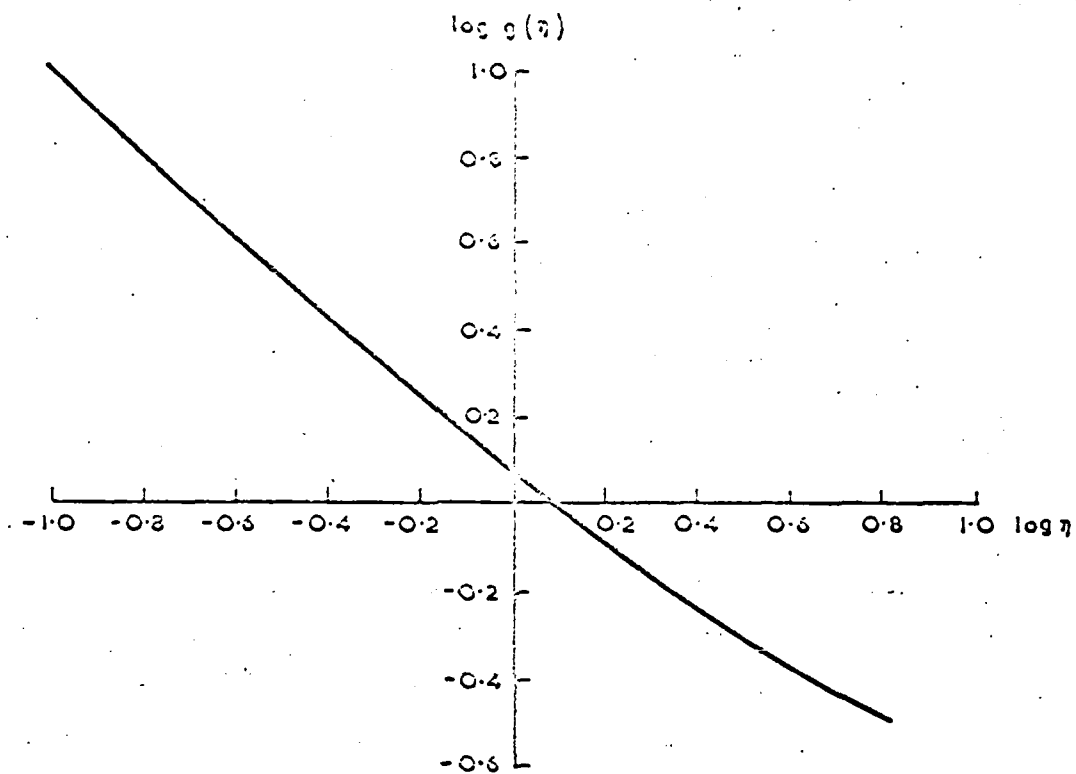
FIGURE 4. THE COEFFICIENT $c(\eta)$ IN EQUATION (26)